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The Chiral Coupling Constants $\bar{l}_1$ and $\bar{l}_2$ from $\pi\pi$ Phase Shifts

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Abstract

A Roy equation analysis of the available $\pi\pi$ phase shift data is performed with the $I = 0$ S- wave scattering length a_0^0 in the range predicted by the one-loop standard chiral perturbation theory. A suitable dispersive framework is developed to extract the chiral coupling constants $\bar{l}_1$, $\bar{l}_2$ and yields $\bar{l}_1 = -1.70 \pm 0.15$ and $\bar{l}_2 \approx 5.0$. We remark on the implications of this determination to (combinations of) threshold parameter predictions of the three lowest partial waves.

1 Introduction

Chiral perturbation theory [1, 2, 3] provides the low energy effective theory of the standard model that describes interactions involving hadronic degrees of freedom and exploits the near masslessness of the u, d (and s) quarks and the observation that the pions, kaons and the η could be viewed as the Goldstone bosons of the spontaneously broken axial symmetry of massless QCD. It is a non-renormalizable theory and involves additional coupling constants that have to be introduced at each order in the derivative or momentum expansion. [From here on we confine our attention to the $SU(2)$ flavor subgroup.] At leading order, $O(p^2)$, we have the pion decay constant, $F_\pi \simeq 93$ MeV in addition to the mass of the pion, $m_\pi = 139.6$ MeV, henceforth set equal to 1. As a result, one has for what is arguably the simplest purely hadronic process of $\pi\pi$ scattering a prediction for a key threshold parameter, the $I = 0$, S- wave scattering length $a_0^0 = 7/(32\pi F_\pi^2) \simeq 0.16$ [4]. There are ten more coupling constants at the next to leading order, $O(p^4)$; four of them, $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$ and $\bar{l}_4$ enter the $\pi\pi$ scattering amplitude [1]. As a result, at this order a_0^0 (and a_0^2) are predicted in terms of these as well. One of the cornerstones of standard chiral perturbation theory at $O(p^4)$ is a relatively definite prediction of a_0^0 in the range 0.19-0.21.

The coupling constants $\bar{l}_1$ and $\bar{l}_2$ have been fixed in the past from experimental values available in the literature [5] for the D- wave scattering

lengths:

$$a_2^0 = 17 \pm 3 \cdot 10^{-4}, \quad a_2^2 = 1.3 \pm 3 \cdot 10^{-4}.$$

The one-loop expressions for these are [2]:

$$a_2^0 = \frac{1}{1440\pi^3 F_\pi^4}(\bar{l}_1 + 4\bar{l}_2 - \frac{53}{8}), \quad a_2^2 = \frac{1}{1440\pi^3 F_\pi^4}(\bar{l}_1 + \bar{l}_2 - \frac{103}{40}) \quad (1.1)$$

and yield

$$\bar{l}_1 = -2.3 \pm 3.7, \quad \bar{l}_2 = 6.0 \pm 1.3.$$

These have also been determined from analysis of K_{l4} decays [6] which yield

$$\bar{l}_1 = -0.7 \pm 0.9, \quad \bar{l}_2 = 6.3 \pm 0.5$$

and more recently by estimating higher order corrections to these decays [7]

$$\bar{l}_1 = -1.7 \pm 1.0, \quad \bar{l}_2 = 6.1 \pm 0.5.$$

$\bar{l}_1$ and $\bar{l}_2$ are coupling constants consistent with the presence of resonances. In particular the ρ resonance may make a significant contribution [see Appendix C in Ref. [2]] and has also been discussed more recently [8]. Furthermore, generalized vector meson dominance [9] leads to numerical values for these consistent with the numbers above. Tensor resonances also have been found to make non-trivial contributions [10].

The constants $\bar{l}_3$ has been estimated from the analysis of $SU(3)$ mass relations which yields [2]:

$$\bar{l}_3 = 2.9 \pm 2.4.$$

The variation of a_0^0 is essentially equivalent to the variation of $\bar{l}_3$. While here $\bar{l}_3$ would have to be -70 in order to accommodate $a_0^0 = 0.26$, there is an extended framework which re-orders the chiral power counting in order to accommodate such large values of a_0^0 modifying even the tree-level prediction [11]. Here we confine ourselves to the more predictive standard chiral perturbation theory whose stringent predictions will come under experimental scrutiny [12]. Thus we note that in our final analysis we cannot claim an independent determination of $\bar{l}_3$ via the Roy equation analysis of this work since a_0^0 is varied anyway in the range predicted by standard chiral perturbation theory.

The constant $\bar{l}_4$ enters the one-loop expression of the relatively accurately determined value of the "universal curve" combination $2a_0^0 - 5a_0^2$ [13] and is also related to the independent estimates of the scalar charge radius of the pion. An $SU(3)$ analysis that has been performed for the ratio of the kaon and pion decay constants F_K/F_π also provides a measure of this constant $\bar{l}_4 \approx 4.6$ [14]. In the present analysis we treat a_0^0 as the only free parameter to the fit to the data and a_0^2 is produced as an output corresponding to the optimal solution of our data fitting. In particular, the values we find remain consistent with the universal-curve band. Thus we have a determination of the constant $\bar{l}_4$. However, we also perform constrained fits with a_0^2 computed from certain universal curve relations that fix $\bar{l}_4$ a priori. The influence on the actual numerical fits is found to be minimal reflecting the weakness of

the $I = 2$ channel and influences the determination of $\bar{l}_1$ and $\bar{l}_2$ minimally due to reasons we discuss in subsequent sections.

On the other hand $\pi\pi$ scattering has been studied in great detail in axiomatic field theory [15]. (Fixed-t) dispersion relations with two subtractions, a number dictated by the Froissart bound, have been rigorously established in the axiomatic framework. The properties of crossing and analyticity have been exploited in order to establish the Roy equations [16, 17], a system of integral equations for the partial waves. The Roy equations have been the basis of analysis of phase shift data [18] and a knowledge of the threshold parameters involved in $\pi\pi$ scattering has been obtained. Best fits to Roy equation analysis of data are obtained with $a_0^0 = 0.26 \pm 0.05$ [19]. Note that the D- wave scattering lengths cited earlier have also been extracted from Roy equation analysis. The properties of analyticity, unitarity and crossing and positivity of absorptive parts have also been shown to produce non-trivial constraints on the magnitudes of a certain combination of $\bar{l}_1$ and $\bar{l}_2$[20].

Here we report on a direct determination of the chiral coupling constants from the existing phase shift data itself by performing a Roy equation fit to it when a_0^0 is confined to the range predicted by chiral perturbation theory. The chiral amplitude is rewritten in terms of a dispersive representation with a certain number of effective subtractions consistent with $O(p^4)$ accuracy, where the subtraction constants are expressed in terms of the chiral coupling constants. The fixed-t dispersion relations of axiomatic field theory are also

rewritten in a form whereby a direct comparison can be made with the chiral dispersive representation, while the effective subtraction constants are now computed in terms of physical partial waves produced by the Roy equation fit, the input value of a_0^0 and the resulting value of a_0^2 generated by the fit. In most of our treatment we limit ourselves to a certain approximation where we account for the absorptive parts of $l \geq 2$ states only through the "driving terms" of the Roy equations for the S- and P- waves.

Furthermore, we also perform an analysis of certain threshold parameters computed from the Roy equation fits which reveals the magnitude of $O(p^6)$ corrections their one-loop predictions are expected to suffer from. The dispersive framework can be extended to meet the needs of two-loop chiral perturbation theory and used to pin down the associated coupling constants [21, 22]. The work reported here summarizes the first stage of our computations and is presently being extended to meet the needs of the two-loop computation of [22].

2 $\pi\pi$ Scattering to $O(p^4)$ in chiral perturbation theory and the Roy equation solutions

The notation and formalism that we adopt in this discussion follows that of Ref. [18]. Consider $\pi\pi$ scattering:

$$\pi^a(p_a) + \pi^b(p_b) \to \pi^c(p_c) + \pi^d(p_d),$$

where all the pions have the same mass. The Mandelstam variables s, t and u are defined as

$$s = (p_a + p_b)^2, \quad t = (p_a - p_c)^2, \quad t = (p_a - p_d)^2, \quad s + t + u = 4. \tag{2.1}$$

The scattering amplitude $F(a, b \to c, d)$ (our normalization of the amplitude differs from that of Ref. [18] by 32π and is consistent with that of Ref. [2, 23]) can then be written as

$$F(a, b \to c, d) = \delta_{ab}\delta_{cd}A(s, t, u) + \delta_{ac}\delta_{bd}A(t, s, u) + \delta_{ad}\delta_{bc}A(u, t, s).$$

From $A(s, t, u)$ we construct the three s-channel isospin amplitudes:

$$\begin{aligned} T^0(s, t, u) &= 3A(s, t, u) + A(t, s, u) + A(u, t, s), \\ T^1(s, t, u) &= A(t, s, u) - A(u, t, s), \\ T^2(s, t, u) &= A(t, s, u) + A(u, t, s). \end{aligned} \tag{2.2}$$

We introduce the partial wave expansion:

$$T^I(s, t, u) = 32\pi \sum_{l=0}^{\infty} (2l + 1) P_l\left(\frac{t - u}{s - 4}\right) f_l^I(s), \tag{2.3}$$

$$f_l^0(s) = f_l^2(s) = 0, l \text{ odd}, \quad f_l^1(s) = 0, l \text{ even}.$$

The unitarity condition for the partial wave amplitudes $f_l^I(s)$ is:

$$\mathrm{Im} f_l^I(s) = \rho(s)|f_l^I(s)|^2 + \frac{1-(\eta_l^I(s))^2}{4\rho(s)}, \tag{2.4}$$

where $\rho(s) = \sqrt{(s-4)/s}$ and $\eta_l^I(s)$ is the elasticity at a given energy $\sqrt{s}$. We also introduce the threshold expansion:

$$\mathrm{Re} f_l^I(s) = \left(\frac{s-4}{4}\right)^l \left(a_l^I + b_l^I \left(\frac{s-4}{4}\right) + \ldots\right), \tag{2.5}$$

where the a_l^I are the scattering lengths and the b_l^I are the effective ranges, namely the leading threshold parameters.

Chiral perturbation theory at next to leading order gives an explicit representation for the function $A(s,t,u)$ at $O(p^4)$ [2]:

$$A(s,t,u) = A^{(2)}(s,t,u) + A^{(4)}(s,t,u) + O(p^6), \tag{2.6}$$

with

$$\begin{aligned}
A^{(2)}(s,t,u) &= \frac{s-1}{F_\pi^2}, \\
A^{(4)}(s,t,u) &= \frac{1}{6F_\pi^4}\Big(3(s^2-1)\bar{J}(s) \\
&\quad + [t(t-u)-2t+4u-2]\bar{J}(t) + (t \leftrightarrow u)\Big) \\
&\quad + \frac{1}{96\pi^2 F_\pi^4}\{2(\bar{l}_1 - 4/3)(s-2)^2 + (\bar{l}_2 - 5/6)[s^2 + (t-u)^2] \\
&\quad + 12s(\bar{l}_4 - 1) - 3(\bar{l}_3 + 4\bar{l}_4 - 5)\} \\
\text{and } \bar{J}(z) &= -\frac{1}{16\pi^2}\int_0^1 dx \ln[1 - x(1-x)z], \quad \mathrm{Im}\bar{J}(z) = \frac{\rho(s)}{16\pi}\Theta(z-4).
\end{aligned}$$

Note also that at $O(p^4)$ the imaginary parts of the partial waves above threshold computed ($s > 4$) from the amplitude above is:

$$\begin{aligned}
\mathrm{Im} f_0^0(s) &= \frac{\rho(s)}{1024\pi^2 {F_\pi}^4}(2s-1)^2 \\
\mathrm{Im} f_1^1(s) &= \frac{\rho(s)}{9216\pi^2 {F_\pi}^4}(s-4)^2 \\
\mathrm{Im} f_0^2(s) &= \frac{\rho(s)}{1024\pi^2 {F_\pi}^4}(s-2)^2 \qquad (2.7) \\
\mathrm{Im} f_l^I(s) &= 0, \quad l \geq 2.
\end{aligned}$$

[Note that the chiral power counting enforces the property that the absorptive parts of the D- and higher waves arise only at $O(p^8)$.] Furthermore these verify the property of perturbative unitarity, viz., when the $O(p^2)$ predictions for the threshold parameters $a_0^0 = 7/(32\pi {F_\pi}^2)$, $a_0^2 = -1/(16\pi {F_\pi}^2)$, $b_0^0 = 1/(4\pi {F_\pi}^2)$, $b_0^2 = -1/(8\pi {F_\pi}^2)$ and $a_1^1 = 1/(24\pi {F_\pi}^2)$ are inserted into the pertinent form of the perturbative unitarity relations:

$$\begin{aligned}
\mathrm{Im} f_0^I(s) &= \rho(s)(a_0^I + b_0^I(s-4)/4)^2, \ I = 0, 2 \\
\mathrm{Im} f_1^1(s) &= \rho(s)(a_1^1(s-4)/4)^2.
\end{aligned}$$

In order to carry out the comparison between the chiral expansion and the physical scattering data, we first recall that up to $O(p^6)$, it is possible to decompose $A(s,t,u)$ into a sum of three functions of single variables as follows [24]:

$$A(s,t,u) = 32\pi\left[\frac{1}{3}W_0(s) + \frac{3}{2}(s-u)W_1(t) + \frac{3}{2}(s-t)W_1(u) + \frac{1}{2}\left(W_2(t) + W_2(u) - \frac{2}{3}W_2(s)\right)\right]. \tag{2.8}$$

One convenient decomposition of the chiral one-loop amplitude is:

$$\begin{aligned} W_0(s) &= \frac{3}{32\pi}\left[\frac{s-1}{F_\pi^2} + \frac{2}{3F_\pi^4}(s-1/2)^2\bar{J}(s) \right. \\ &\quad + \frac{1}{96\pi^2 F_\pi^4}(2(\bar{l}_1 - 4/3)(s-2)^2 + 4/3(\bar{l}_2 - 5/6)(s-2)^2 \\ &\quad \left. + 12s(\bar{l}_4 - 1) - 3(\bar{l}_3 + 4\bar{l}_4 - 5))\right], \end{aligned} \tag{2.9}$$

$$W_1(s) = \frac{1}{576\pi F_\pi^4}(s-4)\bar{J}(s), \tag{2.10}$$

$$W_2(s) = \frac{1}{16\pi}\left[\frac{1}{4F_\pi^4}(s-2)^2\bar{J}(s) + \frac{1}{48\pi^2 F_\pi^4}(\bar{l}_2 - 5/6)(s-2)^2\right], \tag{2.11}$$

where we note that this decomposition is not unique, with ambiguities in the real part only. We observe that the imaginary parts of these functions verify the relation:

$$\mathrm{Im}W_I(x) = \mathrm{Im}f_0^I(x), \quad I = 0,2$$
$$\mathrm{Im}W_1(x) = \mathrm{Im}f_1^1(x)/(x-4),$$

which may be used to demonstrate the following dispersion relations:

$$
\begin{aligned}
W_0(s) &= \frac{-1+72\bar{l}_1+48\bar{l}_2-27\bar{l}_3-108\bar{l}_4-864\pi^2{F_\pi}^2}{9216\pi^3{F_\pi}^4} \\
&\quad+\frac{59-144\bar{l}_1-96\bar{l}_2+216\bar{l}_4+1728\pi^2{F_\pi}^2}{18432\pi^3{F_\pi}^4}s \qquad (2.12)\\
&\quad+\frac{-797+360\bar{l}_1+240\bar{l}_2}{184320\pi^3{F_\pi}^4}s^2+\frac{s^3}{\pi}\int_4^\infty\frac{dx}{x^3(x-s)}\mathrm{Im}f_0^0(x),
\end{aligned}
$$

$$
W_1(s) = \frac{-s}{13824\pi^3{F_\pi}^4}+\frac{s^2}{\pi}\int_4^\infty\frac{dx}{x^2(x-4)(x-s)}\mathrm{Im}f_1^1(x), \qquad (2.13)
$$

$$
\begin{aligned}
W_2(s) &= \frac{6\bar{l}_2-5}{1152\pi^3{F_\pi}^4}+\frac{23-24\bar{l}_2}{4608\pi^3{F_\pi}^4}s+\frac{60\bar{l}_2-77}{46080\pi^3{F_\pi}^4}s^2 \qquad (2.14)\\
&\quad+\frac{s^3}{\pi}\int_4^\infty\frac{dx}{x^3(x-s)}\mathrm{Im}f_0^2(x).
\end{aligned}
$$

We now reconstruct $A(s,t,u)$ from this dispersive representation for the W's to obtain:

$$
\begin{aligned}
A(s,t,u) &= \frac{s-1}{{F_\pi}^2}+\frac{-540+480\bar{l}_1+960\bar{l}_2-180\bar{l}_3-720\bar{l}_4}{5760\pi^2{F_\pi}^4}\\
&\quad-\frac{110+480\bar{l}_1-720\bar{l}_4}{5760\pi^2{F_\pi}^4}s-\frac{163-120\bar{l}_1}{5760\pi^2{F_\pi}^4}s^2\\
&\quad+\frac{460-480\bar{l}_2}{5760\pi^2{F_\pi}^4}(t+u)-\frac{20u(s-t)+20t(s-u)}{5760\pi^2{F_\pi}^4}\\
&\quad-\frac{154-120\bar{l}_2}{5760\pi^2{F_\pi}^4}(t^2+u^2) \qquad (2.15)\\
&\quad+32\pi\left(\frac{1}{3}\frac{s^3}{\pi}\int_4^\infty\frac{dx}{x^3(x-s)}\left(\mathrm{Im}f_0^0(x)-\mathrm{Im}f_0^2(x)\right)\right.\\
&\quad+\frac{3}{2}(s-u)\frac{t^2}{\pi}\int_4^\infty\frac{dx}{x^2(x-t)(x-4)}\mathrm{Im}f_1^1(x)\\
&\quad+\frac{3}{2}(s-t)\frac{u^2}{\pi}\int_4^\infty\frac{dx}{x^2(x-u)(x-4)}\mathrm{Im}f_1^1(x)\\
&\quad\left.+\frac{1}{2}\left(\frac{t^3}{\pi}\int_4^\infty\frac{dx}{x^3(x-t)}\mathrm{Im}f_0^2(x)+\frac{u^3}{\pi}\int_4^\infty\frac{dx}{x^3(x-u)}\mathrm{Im}f_0^2(x)\right)\right).
\end{aligned}
$$

This is seen to be the sum of a polynomial of second degree in s, t and u and a dispersive piece. The problem associated with the non-uniqueness of the real part of the decomposition into the W's is eliminated by setting $u = 4 - s - t$ upon which we obtain a second degree polynomial in s and t:

$$
\begin{aligned}
P = &\left(\frac{29}{120\pi^2 {F_\pi}^4} - \frac{\bar{l}_2}{6\pi^2 {F_\pi}^4}\right)\left(t - \frac{t^2}{4} - \frac{st}{4}\right) \\
&+ \left(-\frac{33}{640\pi^2 {F_\pi}^4} + \frac{\bar{l}_1}{48\pi^2 {F_\pi}^4} + \frac{\bar{l}_2}{48\pi^2 {F_\pi}^4}\right) s^2 \\
&+ \left(\frac{1}{{F_\pi}^2} + \frac{97}{960\pi^2 {F_\pi}^4} - \frac{\bar{l}_1}{12\pi^2 {F_\pi}^4} - \frac{\bar{l}_2}{12\pi^2 {F_\pi}^4} + \frac{\bar{l}_4}{8\pi^2 {F_\pi}^4}\right) s \\
&+ \left(-\frac{1}{{F_\pi}^2} - \frac{97}{480\pi^2 {F_\pi}^4} + \frac{\bar{l}_1}{12\pi^2 {F_\pi}^4} + \frac{\bar{l}_2}{6\pi^2 {F_\pi}^4} - \frac{\bar{l}_3}{32\pi^2 {F_\pi}^4} - \frac{\bar{l}_4}{8\pi^2 {F_\pi}^4}\right).
\end{aligned}
\tag{2.16}
$$

The Roy equation fit allows us to obtain a representation only for the S- and P- wave absorptive parts, [with some effects of higher angular momentum states absorbed into the driving terms (see Appendix A)]. Thus, a determination of the physical S- and P- wave absorptive parts, allows us to construct a set of crossing symmetric amplitudes [17, 25] from which we extract a representation for $A(s,t,u)$ (see Appendix A for details):

$$
\begin{aligned}
A(s,t,u) = &\frac{32\pi}{3}(\gamma_0^0 - \gamma_0^2)s^2 + 16\pi\gamma_0^2(t^2+u^2) + 4\pi\alpha_0^2(u+t) \\
&+\frac{8\pi}{3}(\alpha_0^0 - \alpha_0^2)s + 16\pi\beta_1^1 t(s-u) + 16\pi\beta_1^1 u(s-t) \\
&+32\pi\left(\frac{1}{3}\frac{s^3}{\pi}\int_4^\infty \frac{dx}{x^3(x-s)}\left(\mathrm{Im} f_0^0(x) - \mathrm{Im} f_0^2(x)\right)\right. \\
&+ \quad \frac{3}{2}(s-u)\frac{t^2}{\pi}\int_4^\infty \frac{dx}{x^2(x-t)(x-4)}\mathrm{Im} f_1^1(x)
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
+ \quad & \frac{3}{2}(s-t)\frac{u^2}{\pi}\int_4^\infty \frac{dx}{x^2(x-u)(x-4)}\mathrm{Im}f_1^1(x) \\
+ \quad & \frac{1}{2}\left(\frac{t^3}{\pi}\int_4^\infty \frac{dx}{x^3(x-t)}\mathrm{Im}f_0^2(x) + \frac{u^3}{\pi}\int_4^\infty \frac{dx}{x^3(x-u)}\mathrm{Im}f_0^2(x)\right)\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_0^I &= a_0^I - \frac{4}{\pi}\int_4^\infty \frac{dx}{x(x-4)}\mathrm{Im}f_0^I(x) + \frac{4}{\pi}\int_4^\infty \frac{dx}{x^2}\mathrm{Im}f_0^I(x) \quad I=0,2 \\
\gamma_0^I &= \frac{1}{\pi}\int_4^\infty \frac{dx}{x^3}\mathrm{Im}f_0^I(x) \quad I=0,2 \\
\beta_1^1 &= \frac{3}{\pi}\int_4^\infty \frac{dx}{x^2(x-4)}\mathrm{Im}f_1^1(x) \\
\alpha_0^1 &= \beta_1^0 = \beta_1^2 = 0.
\end{aligned} \tag{2.18}
$$

We are now in a position to compare the two representations for $A(s,t,u)$ namely the chiral representation eq.(2.15) and the axiomatic representation eq.(2.17). These are formally equivalent, with the dispersive integrals in the former described by chiral absorptive parts whereas in the latter by the physical S- and P-wave absorptive parts. For the chiral expansion to reproduce low energy physics accurately we now require the effective subtraction constants to match. Once more setting $u = 4-s-t$ yields the polynomial piece of the representation eq.(2.17):

$$
\begin{aligned}
P = & -128\pi(\beta_1^1+\gamma_0^2)(t-\frac{t^2}{4}-\frac{st}{4}) + \frac{16\pi}{3}(2\gamma_0^0+\gamma_0^2-3\beta_1^1)s^2 \\
& +8\pi(\frac{\alpha_0^0}{3}-\frac{5}{6}\alpha_0^2+8\beta_1^1-16\gamma_0^2)s + 16\pi(\alpha_0^2+16\gamma_0^2).
\end{aligned} \tag{2.19}
$$

A straightforward comparison of eq. (2.16) and eq. (2.19) yields explicit

expressions for $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$ and $\bar{l}_4$. In particular we have for $\bar{l}_1$ and $\bar{l}_2$:

$$\bar{l}_1 = 24\pi^2 {F_\pi}^4 \left(\frac{41}{960\pi^2 {F_\pi}^4} - \frac{64\pi}{3}(\gamma_0^2 - \gamma_0^0 + 3\beta_1^1)\right), \tag{2.20}$$

$$\bar{l}_2 = 24\pi^2 {F_\pi}^4 \left(\frac{29}{480\pi^2 {F_\pi}^4} + 32\pi(\beta_1^1 + \gamma_0^2)\right). \tag{2.21}$$

The actual numerical values we find for $\bar{l}_1$ and $\bar{l}_2$ are reported in a subsequent section. These have an interesting dependence on the actual physical phase shifts: one observes that in eq.(2.20), the presence of γ_0^0. As a result we can anticipate $\bar{l}_1$ to be influenced by the input for a_0^0. In contrast, $\bar{l}_2$ has no dependence on γ_0^0 and depends almost totally on the P- wave contribution via β_1^1, as a result of the weakness of the $I = 2$ channel which renders γ_0^2 negligible in comparison with β_1^1 (and γ_0^0). Since the P- wave happens to be the best determined experimental quantity, even the Roy equation fits to it are not strongly influenced by the input value of a_0^0. Thus we expect a determination of $\bar{l}_2$ in this manner to be very stable.

The values we find for $\bar{l}_3$ and $\bar{l}_4$ from the procedure above are in rough agreement with the estimates provided in the introduction, but suffer from the fact that here the comparison also involves the $O(p^2)$; they are determined only after large cancelations occur and thus a determination of these are not expected to be very reliable. As noted earlier the determination in this manner of $\bar{l}_3$ cannot be considered independent of the input a_0^0 and so we do not report it here. $\bar{l}_4$ may be determined from the dispersive formulas here.

3 Implications to Some Threshold Parameters

Specific combinations of threshold parameters appear on the left hand side when Roy equations for some of the lowest partial waves (and for instance their energy derivatives) are evaluated at threshold. On the right hand sides one has energy integrals over partial wave absorptive parts. These serve as sum rules for such combinations. Indeed expressions for these sum rules have been derived even before the Roy equation program (see e.g., [26]). While there are several inequivalent methods of obtaining such sum rules, it has been shown that as long as one is confined to the absorptive parts of the S- and P- waves alone, each of these methods would yield identical results for the right hand sides (for a recent discussion, see [27]).

We consider the Roy equations eq. (A.3) in the following limits:

$$\lim_{s\to 4+} \frac{d}{ds}\left(12\mathrm{Re}f_0^0(s)\right), \quad \lim_{s\to 4+} \frac{d}{ds}\left(24\mathrm{Re}f_0^2(s)\right)$$
$$\lim_{s\to 4+} \frac{18\mathrm{Re}f_1^1(s)}{(s-4)/4}, \quad \lim_{s\to 4+} \frac{d}{ds}\left(\frac{72\mathrm{Re}f_1^1(s)}{(s-4)/4}\right). \tag{3.1}$$

These may then be rearranged to yield the specific combinations of threshold parameters satisfying:

$$3b_0^0 - (2a_0^0 - 5a_0^2) = \frac{16}{\pi}\int_4^\Lambda \frac{dx}{(x(x-4))^2}$$
$$\Big\{4(x-1)\mathrm{Im}f_0^0(x) - 3\frac{x-2}{2}\sqrt{x(x-4)}(a_0^0)^2 \tag{3.2}$$

$$
\begin{aligned}
&-9(x-4)\mathrm{Im}f_1^1(x)+5(x-4)\mathrm{Im}f_0^2(x)\Big\}\\
&+12\lim_{s\to 4+}\frac{d}{ds}\mathrm{Re}\ d_0^0(s,\Lambda),\\
6b_0^2+(2a_0^0-5a_0^2)&=\frac{16}{\pi}\int_4^\Lambda\frac{dx}{(x(x-4))^2}\\
&\Big\{2(x-4)\mathrm{Im}f_0^0(x)+9(x-4)\mathrm{Im}f_1^1(x) \qquad (3.3)\\
&+(7x-4)\mathrm{Im}f_0^2(x)-3(x-2)\sqrt{x(x-4)}(a_0^2)^2\Big\}\\
&+24\lim_{s\to 4+}\frac{d}{ds}\mathrm{Re}\ d_0^2(s,\Lambda),\\
18a_1^1-(2a_0^0-5a_0^2)&=\frac{16}{\pi}\int_4^\Lambda\frac{dx}{(x(x-4))^2}\\
&\Big\{-2(x-4)\mathrm{Im}f_0^0(s)+9(3x-4)\mathrm{Im}f_1^1(x)\\
&+5(x-4)\mathrm{Im}f_0^2(x)\Big\}+18\lim_{s\to 4+}\frac{\mathrm{Re}\ d_1^1(s,\Lambda)}{(s-4)/4}, \qquad (3.4)\\
18b_1^1&=\frac{16}{\pi}\int_4^\Lambda\frac{dx}{(x(x-4))^3}\\
&\Big\{-2(x-4)^3\mathrm{Im}f_0^0(x)+9(3x^3-12x^2+48x-64)\mathrm{Im}f_1^1(x)\\
&+5(x-4)^3\mathrm{Im}f_0^2(x)\Big\}+72\lim_{s\to 4+}\frac{d}{ds}\left(\frac{\mathrm{Re}\ d_1^1(s,\Lambda)}{(s-4)/4}\right). \qquad (3.5)
\end{aligned}
$$

From the following limits for the Roy equations:

$$\lim_{s\to 4+}\frac{\mathrm{Re}f_2^0(s)}{((s-4)/4)^2},\quad \lim_{s\to 4+}\frac{\mathrm{Re}f_2^2(s)}{((s-4)/4)^2} \qquad (3.6)$$

we find expressions of sum rules for the D-wave scattering lengths.

$$
\begin{aligned}
a_2^0 &= \frac{16}{45\pi}\int_4^\Lambda\frac{dx}{x^3(x-4)}\\
&\Big\{(x-4)\mathrm{Im}f_0^0(x)+9(x+4)\mathrm{Im}f_1^1(x)+5(x-4)\mathrm{Im}f_0^2(x)\Big\} \qquad (3.7)\\
&+\lim_{s\to 4+}\frac{\mathrm{Re}\ d_2^0(s,\Lambda)}{((s-4)/4)^2},
\end{aligned}
$$

$$
\begin{aligned}
a_2^2 &= \frac{16}{90\pi}\int_4^\Lambda \frac{dx}{x^3(x-4)} \\
&\Big\{2(x-4)\mathrm{Im}f_0^0(x) - 9(x+4)\mathrm{Im}f_1^1(x) + (x-4)\mathrm{Im}f_0^2(x)\Big\} \qquad (3.8)\\
&+ \lim_{s\to 4+}\frac{\mathrm{Re}\ d_2^2(s,\Lambda)}{((s-4)/4)^2}.
\end{aligned}
$$

The one-loop expressions for the combinations of interest are

$$
3b_0^0 - (2a_0^0 - 5a_0^2) = \frac{1}{16\pi^3 F_\pi^4}(2\bar{l}_1 + 3\bar{l}_2 - \frac{7}{4}), \tag{3.9}
$$

$$
6b_0^2 + (2a_0^0 - 5a_0^2) = \frac{1}{16\pi^3 F_\pi^4}(\bar{l}_1 + 3\bar{l}_2 + \frac{5}{8}), \tag{3.10}
$$

$$
18a_1^1 - (2a_0^0 - 5a_0^2) = \frac{1}{16\pi^3 F_\pi^4}(\bar{l}_2 - \bar{l}_1 - \frac{55}{24}), \tag{3.11}
$$

$$
18b_1^1 = \frac{1}{16\pi^3 F_\pi^4}(\bar{l}_2 - \bar{l}_1 + \frac{97}{120}) \tag{3.12}
$$

respectively. Those for the D-wave scattering lengths are given in eq.(1.1). Since the Roy equations are a consequence of dispersion relations with two subtractions, we note that in all the above the leading $O(p^2)$ contribution cancels exactly. Furthermore, the constants $\bar{l}_3$ and $\bar{l}_4$ are also absent since they accompany only constant and linear pieces in s, t and u in the $O(p^4)$ scattering amplitude. It would therefore seem that any two of the six combinations above is suitable for determination of $\bar{l}_1$ and $\bar{l}_2$ since they enjoy the same status as the D-wave scattering lengths (see eq. (1.1)).

A careful consideration of these reveals some interesting characteristics. Note for instance that the one-loop expressions for $18a_1^1 - (2a_0^0 - 5a_0^2)$ and for $18b_1^1$ depends only on the combination $\bar{l}_2 - \bar{l}_1$. If we now evaluate their numerical values for each of our Roy equation phase shift representations by

inserting them into the sum rules in order to compute the coupling constants of interest we would not have a meaningful result since each of the combinations above represents parallel straight lines in the $\bar{l}_1,\ \bar{l}_2$ plane! If we consider the further combination $18a_1^1 - (2a_0^0 - 5a_0^2) - 18b_1^1$, its one-loop expression is $-31/160\pi^3 {F_\pi}^4$. If we saturate the sum rule for this combination with $\Lambda = \infty$ and the chiral absorptive parts eq.(2.7) we reproduce the one-loop result which is a result of the perturbative unitarity of the chiral expansion. As a result when the sum rule for this combination is evaluated with the physical absorptive parts, we get an answer that is substantially different from its one-loop expression. From this we conclude that the two-loop corrections to this combination must account for this discrepancy, although we cannot conclude which one of the elements in the combination receives the correction. An analogous exercise may be performed for the $\pi^0\pi^0$ combinations and in particular for $30a_2 - b_0$ where $a_2 = (a_2^0 + 2a_2^2)/3$ and $b_0 = (b_0^0 + 2b_0^2)/3$. A variety of combinations that arise from totally symmetric amplitudes that have similar properties has been examined recently [27].

A final exercise we perform is to compute the value for $2a_0^0 - 5a_0^2$ from the Roy equation fit and then to insert the value of $\bar{l}_4$ we find into the one-loop formula for the same combination:

$$2a_0^0 - 5a_0^2 \;=\; \frac{3}{4\pi {F_\pi}^2}\left(1 + \frac{1}{8\pi^2 {F_\pi}^2}(\bar{l}_4 + 5/8)\right).$$

Once again we observe that the agreement, while being fair is not exact

reflecting the somewhat large uncertainties in our determination referred to earlier as well as due to the $O(p^6)$ corrections to its one-loop formula.

4 Numerical Results and Discussion

The numerical solutions of the Roy equations are obtained by using a parameterization similar to the one described in great detail in [18]. In this study we have employed the $\pi\pi$ scattering from the CERN-Munich experiment and documented by Ochs [28] in the region $19 \leq s \leq 60$, and the high precision K_{l4} experimental data [29] for the phase shift difference $\delta_0^0 - \delta_1^1$. We have devised some checks on our computation in the following manner: we have first of all determined the best fit to the parameters of the parameterization referred to above by requiring it to simultaneously yield the best fit to the data as well as satisfy the Roy equations in the domain of their validity. The same parameterization is considered to be valid in the domain $60 \leq s \leq 110$ as well. Then we have employed the Ochs data in this region and have required a best fit to the data in $4 \leq s \leq 110$ and that it satisfies the Roy equation in $4 \leq s \leq 60$. In practice the data above 60 do not influence the parameters of the fit significantly: this in turn renders our numerical results rather stable since many of the quantities we compute are dominated by the low-energy behavior. The details of our work will be documented elsewhere [30].

Our solutions require as an input parameter only a_0^0 and numerically

search for those solutions that minimize the discrepancy with respect to the data. In Fig. 1 we present our phase shift fits for $\delta_0^0 - \delta_1^1$ obtained for $a_0^0 = 0.19$, 0.20, 0.21 and also for the central value $a_0^0 = 0.26$. The final result of the procedure above is an explicit representation for the lowest partial waves as functions of the energy in terms of a few parameters that optimally fit the data up to $\Lambda = 110$ and verifying the Roy equations in their domain of validity.

The numerical values of the α_0^I, γ_0^I, $I = 0, 2$ and β_1^1 may now be computed from the explicit phase shift representation provided by the Roy equation fit, and $\bar{l}_1$ and $\bar{l}_2$ are extracted from eq. (2.20) and (2.21) (the effects arising from the neglect above the cutoff Λ are disregarded; an unrealistic absolute saturation of the integrands above Λ can lower $\bar{l}_1$ by 0.13 and increase $\bar{l}_2$ by the same amount), and $\bar{l}_4$ from the appropriate comparison of eq.(2.16) and eq.(2.19).

The Roy equation fits are also used to compute the combinations of S- and P-wave scattering lengths and effective ranges of eq.(3.2)-(3.5) and the high energy tail and higher wave contributions computed from the driving terms as expressed in them. For the right hand sides of the sum rules for the D- wave scattering lengths, the driving terms for the Roy equations for the D- waves is not available in the literature. The S- and P- wave contributions are explicitly accounted for up to the cut-off and their high energy tail is disregarded in our tabulation of the results (an absolute (unrealistic) saturation of the

integrands above $\Lambda = 110$ yields an error of $+0.7 \times 10^{-4}$ on a_2^0 and an error of $+0.07 \times 10^{-4}$ and -0.22×10^{-4} on a_2^2). The only higher wave contribution arises from the $f_2(1270)$ [31] and we have estimated its contribution from two inequivalent sets of sum rules for these available in the literature [26, 27] which give almost identical results of 0.54×10^{-4} for a_2^0 and 0.38×10^{-4} for a_2^2 and have been included in the final results.

We tabulate our results in Table 1 and 2. In Table 1 for a given input of a_0^0 we report the results of our fit for a_0^2, b_0^0, b_0^2, a_1^1, b_1^1, where the last four are obtained from the sum rules of the previous section and the values of $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_4$ computed from the dispersive relations. In Table 2 (a), 2 (b) and 2 (c) we give values for the 9 combinations of threshold parameters of interest computed from the sum rules and the Roy equation representation and also their one-loop values obtained by inserting the values of $\bar{l}_1$ and $\bar{l}_2$ from the dispersive analysis.

An inspection of this table reveals that the values of the combinations for $18a_1^1 - (2a_0^0 - 5a_0^2)$ in both columns agree better than the values in the two columns for $18b_1^1$. Such an agreement is also seen to be better for a_2 than it is for b_0. We conclude therefore that the determination of $\bar{l}_1$ and $\bar{l}_2$ from the dispersive framework is better consistent with the one-loop expressions for the scattering lengths than it is for the effective ranges.

As a final check we have compared our results obtained from the Roy equation representation with those obtained from a simple analytic param-

eterization of the type proposed by Schenk [32] for the lowest wave phase shifts:

$$\begin{aligned}
\tan\delta_0^I(s) &= \rho(s)\left\{a_0^I + [b_0^I - \frac{a_0^I}{(s_I-4)/4} + (a_0^I)^3](s-4)/4\right\}\frac{s_I-4}{s_I-s},\ I=0,2 \\
\tan\delta_1^1(s) &= \rho(s)\frac{s-4}{4}\{a_1^1 + [b_1^1 - \frac{a_1^1}{(s_\rho-4)/4}](s-4)/4\}\frac{s_\rho-4}{s_\rho-s}.
\end{aligned}$$

This parameterization employs the postulated normal threshold behavior and the properties of threshold expansion of the real part of the partial waves and elastic unitarity and incorporates features of the $\pi\pi$ interaction such as the position of the ρ, $s_\rho = 30$, and that the $I=0$ S- wave passes through $\pi/2$, $s_0 = 36$ (and an unphysical $s_2 = -16$). The remainder of the required inputs are obtained from the first column in each of the Tables 1, 2 and 3 in addition to the values of a_0^0 and a_0^2 that correspond to these and evaluated the $\alpha_0^I, \gamma_0^I, I=0,2$ and β_1^1 with $\Lambda \approx 50$ (K-$\overline{K}$ threshold) and solved for $\bar{l}_1$ and $\bar{l}_2$. For instance for the entries of Table 2 (b) we find $\bar{l}_1 = -1.72$ and $\bar{l}_2 = 5.2$. Thus this simple parameterization confirms our numerical findings to within a surprising 5% accuracy. However when this parameterization is inserted back into the sum rules for the threshold parameters, the agreement is good only to about 15%.

We now remark on contrasting our determinations of $\bar{l}_1$ and $\bar{l}_2$ with those mentioned in the introduction. Our numbers for $\bar{l}_1$ and $\bar{l}_2$ are comfortably accommodated in the range that was first obtained from D- wave scattering lengths in Ref. [2] and more or less accommodated in the range obtained

from the K_{l4} decays. Our value for $\bar{l}_2$ is significantly lower; the somewhat larger values of the $I = 0$ D- wave scattering length used in Ref. [2] than the numbers we find appear to be the cause. In the present work we have presented a clear correlation between the input value of a_0^0 and the D- wave scattering lengths which points towards a 10% smaller value for a_0^2 than the central value of 17×10^{-4} employed there. Furthermore, we are working in a specific dispersive framework where the one-loop expressions for the effective subtraction constants are altogether likely to suffer some higher order corrections, which we have not attempted to analyze in this work. Similarly the other determinations cited are also susceptible to such corrections and it has not been possible to explain quantitatively what the origins of the discrepancy are.

5 Conclusions

The coupling constants $\bar{l}_1$ and $\bar{l}_2$ of the one loop chiral expansion has been accurately determined at $O(p^4)$ precision from a Roy equation analysis of the existing $\pi\pi$ scattering data with $a_0^0 \in (0.19, 0.21)$ predicted by standard chiral perturbation theory. A suitable dispersive framework is used to effect a comparison between the one loop chiral representation and the Roy equation phase shift representation of the $\pi\pi$ amplitudes to obtain

$$\bar{l}_1 = -1.70 \pm 0.15 \text{ and } \bar{l}_2 \approx 5.0.$$

The result is consistent with the bounds obtained on the combination $\bar{l}_1 + 2\bar{l}_2$ in Ref. [20]. Certain ambiguities in determinations of these at $O(p^4)$ from (combinations of) threshold parameters is discussed. The numerical consistency of one-loop results for those involving certain scattering lengths with the new values of $\bar{l}_1$ and $\bar{l}_2$ and their values computed from the Roy equation fits is superior to the consistency for those that involve effective ranges.

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Note added

After this work was completed we received an e-print [33] dealing with the subject of phase shift data, sum rules and chiral coupling constants at $O(p^6)$. The coupling constants in the first row of Table 4 therein corresponds to $\bar{l}_1 = -1.36$ and $\bar{l}_2 = 5.2$. This may be compared with the coupling constants corresponding to our results for $a_0^0 = 0.20$. We find agreement to within a

few percent with the Roy equation solution while the agreement is very good for $\bar{l}_2$ computed from Schenk's model.

Appendix A

Independent of the dynamics governing the interactions, it has been rigorously established that fixed-t dispersion relations with two subtractions may be written down for the amplitudes of definite isospin in terms of unknown t-dependent subtraction functions:

$$\begin{aligned} T^I(s,t,u) &= \sum_{I'=0}^{2} C_{st}^{II'}(C^{I'}(t) + (s-u)D^{I'}(t)) + \\ & \frac{1}{\pi}\int_4^\infty \frac{dx}{x^2}\left(\frac{s^2}{x-s}\mathbf{I}^{II'} + \frac{u^2}{x-u}C_{su}^{II'}\right) A^{I'}(x,t), \end{aligned} \tag{A.1}$$

where $A^I(x,t)$ is the isospin I s-channel absorptive part, C_{st} and C_{su} are the crossing matrices:

$$C_{st} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}, \quad C_{su} = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

and $\mathbf{I}$ is the identity matrix. Bose symmetry implies: $C^1(t) = D^0(t) = D^2(t) = 0$; the unknown t-dependent functions $C^I(t)$ and $D^I(t)$ may be eliminated, using crossing symmetry, in favor of the S-wave scattering lengths.

The Roy equations are obtained upon projecting the resulting dispersion relation onto partial waves and inserting a partial wave expansion for the

absorptive part. They have been rigorously proved to be valid in the domain $-4 \leq s \leq 60$. These are a system of coupled integral equations for partial wave amplitudes of definite isospin I which are related through crossing symmetry to the absorptive parts of all the partial waves. The Roy equations for the S- and P- waves are [16, 17, 18]:

$$\begin{aligned}
f_0^0(s) &= a_0^0 + (2a_0^0 - 5a_0^2)\frac{s-4}{12} + \sum_{I'=0}^{2}\sum_{l'=0}^{\infty}\int_4^\infty dx K_{00}^{l'I'}(s,x)\mathrm{Im}f_{l'}^{I'}(x), \\
f_1^1(s) &= (2a_0^0 - 5a_0^2)\frac{s-4}{72} + \sum_{I'=0}^{2}\sum_{l'=0}^{\infty}\int_4^\infty dx K_{11}^{l'I'}(s,x)\mathrm{Im}f_{l'}^{I'}(x), \qquad \text{(A.2)} \\
f_0^2(s) &= a_0^2 - (2a_0^0 - 5a_0^2)\frac{s-4}{24} + \sum_{I'=0}^{2}\sum_{l'=0}^{\infty}\int_4^\infty dx K_{20}^{l'I'}(s,x)\mathrm{Im}f_{l'}^{I'}(x)
\end{aligned}$$

and for all the higher partial waves written as:

$$f_l^I(s) = \sum_{I'=0}^{2}\sum_{l'=0}^{\infty}\int_4^\infty dx K_{Il}^{l'I'}(s,x)\mathrm{Im}f_{l'}^{I'}(x), \quad l \geq 2,$$

where $K_{lI}^{l'I'}(s,s')$ are the kernels of the integral equations and have been documented elsewhere [17]. Upon cutting off the integral at a large scale Λ and absorbing the contribution of the high energy tail as well as that of all the higher waves over the entire energy range into the driving terms $d_l^I(s,\Lambda)$ we have:

$$
\begin{aligned}
f_0^0(s) &= a_0^0 + (2a_0^0 - 5a_0^2)\frac{s-4}{12} + \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^\Lambda dx K_{00}^{l'I'}(s,x)\mathrm{Im} f_{l'}^{I'}(x) \\
&\quad + d_0^0(s,\Lambda), \\
f_1^1(s) &= (2a_0^0 - 5a_0^2)\frac{s-4}{72} + \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^\Lambda dx K_{11}^{l'I'}(s,x)\mathrm{Im} f_{l'}^{I'}(x) \\
&\quad + d_1^1(s,\Lambda), \qquad\qquad \text{(A.3)} \\
f_0^2(s) &= a_0^2 - (2a_0^0 - 5a_0^2)\frac{s-4}{24} + \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^\Lambda dx K_{20}^{l'I'}(s,x)\mathrm{Im} f_{l'}^{I'}(x) \\
&\quad + d_0^2(s,\Lambda)
\end{aligned}
$$

and for all the higher partial waves written as:

$$
f_l^I(s) = \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^\Lambda dx K_{Il}^{l'I'}(s,x)\mathrm{Im} f_{l'}^{I'}(x) + d_l^I(s,\Lambda), \quad l \geq 2.
$$

The driving terms themselves for the two lowest partial waves, the $I = 0, 2$ S- waves and the $I = 1$ P- wave are available in the literature when $\Lambda = 110$ [18]. [We note that the driving terms for the $l \geq 2$ partial waves are not documented in the literature.] The primary aim of this work is to derive as completely as possible the information on the S- and P- waves and that of the numerical fit to the experimental data (described in a subsequent section) to provide a parametric representation for the physical S- and P- wave partial waves.

Out of the absorptive parts of the physical S- and P- waves, one may construct a manifestly crossing symmetric amplitudes [17, 25]:

$$\frac{1}{32\pi}T^I(s,t,u) = \sum_{I'=0}^{2}\frac{1}{4}(s\mathbf{I}^{II'} + tC_{st}^{II'} + uC_{su}^{II'})a_0^{I'} + \frac{1}{\pi}\int_4^\infty \frac{dx}{x(x-4)}\cdot \left\{\left[\frac{s(s-4)}{x-s}\mathbf{I}^{II'} + \frac{t(t-4)}{x-t}C_{st}^{II'} + \frac{u(u-4)}{x-u}C_{su}^{II'}\right]\mathrm{Im}f_0^{I'}(x) + 3\left[\frac{s(t-u)}{x-s}\mathbf{I}^{II'} + \frac{t(s-u)}{x-t}C_{st}^{II'} + \frac{u(t-s)}{x-u}C_{su}^{II'}\right]\mathrm{Im}f_1^{I'}(x)\right\}. \tag{A.4}$$

Our objective may be met by first rewriting this dispersion relation as:

$$\frac{1}{32\pi}T^I(s,t,u) = \sum_{I'=0}^{2}\frac{1}{4}(s\mathbf{I}^{II'} + tC_{st}^{II'} + uC_{su}^{II'})\alpha_0^{I'} + (s^2\mathbf{I}^{II'} + t^2C_{st}^{II'} + u^2C_{su}^{II'})\gamma_0^{I'}$$
$$+(s(t-u)\mathbf{I}^{II'} + t(s-u)C_{st}^{II'} + u(t-s)C_{su}^{II'})\beta_1^{I'}$$
$$+\frac{1}{\pi}\int_4^\infty \frac{dx}{x^3}\left(\frac{s^3}{x-s}\mathbf{I}II' + \frac{t^3}{x-t}C_{st}^{II'} + \frac{u^3}{x-u}C_{su}^{II'}\right)\mathrm{Im}f_0^{I'}(x)$$
$$+\frac{3}{\pi}\int_4^\infty \frac{dx}{x^2(x-4)}\left(\frac{s^2(t-u)}{x-s}\mathbf{I}^{II'} + \frac{t^2(s-u)}{x-t}C_{st}^{II'} + \frac{u^2(t-s)}{x-u}C_{su}^{II'}\right)\mathrm{Im}f_1^{I'}(x),$$

where α_0^I, β_1^1 and γ_0^I are defined in eq.(2.18). We now invert the isospin amplitudes eq. (2.2) to obtain $A(s,t,u)$ that is constructed out of the S- and P- wave absorptive parts:

$$A(s,t,u) \equiv \frac{1}{3}\Big(T^0(s,t,u) - T^2(s,t,u)\Big) \tag{A.5}$$

and is given in eq. (2.17).

References

[1] J. Gasser and H. Leutwyler, Phys. Lett. 125B (1983) 325.

[2] J. Gasser and H. Leutwyler, Ann. Phys. (N. Y.) 158 (1984) 142.

[3] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465.

[4] S. Weinberg, Phys.Rev. Lett. 17 (1966) 616; Also see for related results Y. Tomozawa, Nuo. Cim. XLVI A (1966) 707; A. P. Balachandran, M. G. Gundzik and F. Nicodemi, Nuo. Cim. XLIV A (1966) 5773.

[5] M.M. Nagels et al., Nucl. Phys. B147 (1979) 189.

[6] C. Riggenbach, J.F. Donoghue, J. Gasser and B.R. Holstein, Phys.Rev. D 43 (1991) 127.

[7] J. Bijnens, G. Colangelo and J. Gasser, Nucl. Phys. B427 (1994) 427.

[8] M.R. Pennington and J. Portoles, Phys. Lett. B344 (1995) 399.

[9] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl.Phys. B321 (1989) 311.

[10] D. Toublan, UNIL-TP-3/95, hep-ph/9509217 and reference therein.

[11] See e. g., M. Knecht and J. Stern, The Second DAΦNE Physics Handbook, L. Maini, G. Pancheri and N. Paver, eds., INFN Publication (1995), 169.

[12] See e.g., J. Gasser The Second DAΦNE Physics Handbook, L. Maini, G. Pancheri and N. Paver, eds., INFN Publication (1995), 215.

[13] B. R. Martin, D. Morgan and G. Shaw, "Pion-Pion Interaction in Particle Physics," Academic Press, London/New York, 1976.

[14] J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 517.

[15] See, e. g., A. Martin, "Scattering Theory: Unitarity, Analyticity and Crossing," Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[16] S.M. Roy, Phys. Lett. B36 (1971) 353.

[17] J.–L. Basdevant, J.C. Le Guillou and H. Navelet, Nuovo Cimento 7 A (1972) 363.

[18] J.–L. Basdevant, C.D. Froggatt and J.L. Petersen, Nucl. Phys. B 72 (1974) 413.

[19] C.D. Froggatt and J.L. Petersen, Nucl. Phys. B129 (1977) 89.

[20] B. Ananthanarayan, D. Toublan and G. Wanders, Phys. Rev. D51 (1995) 1093.

[21] M. Knecht, B. Mousallam, J. Stern and N. H. Fuchs, Nucl. Phys. B 457 (1995) 513.

[22] J. Bijnens, G. Colangelo,G. Ecker, J. Gasser and M. Sainio, hep-ph/9511397.

[23] J. Gasser and Ulf–G. Meißner, Phys. Lett. B 258 (1991) 219.

[24] J. Stern, H. Sazdjian and N. H. Fuchs, Phys. Rev. D47 (1993) 3814.

[25] G. Mahoux, S.M. Roy and G. Wanders, Nucl. Phys. B70, (1974), 297.

[26] G. Wanders, Helv. Phys. Acta 39 (1966) 228.

[27] B. Ananthanarayan, D. Toublan and G. Wanders, hep-ph/9510254.

[28] W. Ochs, Thesis, Ludwig-Maximilians-Universität, München, 1973.

[29] L. Rosselet, et al., Phys. Rev. D 15 (1977) 574.

[30] P. Büttiker, Ph. D. thesis, Universität Bern (in preparation).

[31] Particle Data Group, L. Montanet et al., Phys. Rev. D 73 (1994) 1173.

[32] A. Schenk, Nucl. Phys. B 363 (1991) 97.

[33] M. Knecht, B. Mousallam, J. Stern and N. H. Fuchs,hep-ph/9512404.

Table Captions

Table 1. The computed values corresponding to the input a_0^0 of a_0^2, b_0^0, b_0^2, a_1^1, b_1^1 and the computed values of $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_4$.

Table 2. (a) Values of combinations of threshold parameters corresponding to the fit of the first line of Table 1 and their one loop values with the new $\bar{l}_1$ and $\bar{l}_2$; (b) As in (a) for the second line; (c) As in (a) for the third line

Figure Caption

Fig. 1. Results of our fit to the combination $\delta_0^0 - \delta_1^1$ (in degrees) as a function of energy (MeV): full line $a_0^0 = 0.26$ dash–dotted line: $a_0^0 = 0.21$, dashed line: $a_0^0 = 0.2$ dash–double–dotted line: $a_0^0 = 0.19$. Also shown is the Rosselet data.

	a_0^0	a_0^2	b_0^0	b_0^2	a_1^1	b_1^1	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_4$
1	0.19	-0.040	0.238	-0.074	0.035	0.006	-1.80	4.98	0.87
2	0.20	-0.037	0.237	-0.074	0.035	0.006	-1.69	4.97	1.09
3	0.21	-0.035	0.238	-0.075	0.035	0.006	-1.58	4.96	1.30

Table 1

	Roy equations	One loop formula
$18a_1^1 - (2a_0^0 - 5a_0^2)$	0.0459	0.0472
$18b_1^1$	0.1079	0.0788
a_2^0	15×10^{-4}	13×10^{-4}
a_2^2	$0.6 \cdot 10^{-4}$	$0.4 \cdot 10^{-4}$
$3b_0^0 - (2a_0^0 - 5a_0^2)$	0.1326	0.0924
$6b_0^2 + (2a_0^0 - 5a_0^2)$	0.1378	0.1370
a_2	0.0005	0.0005
b_0	0.030	0.026
$2a_0^0 - 5a_0^2$	0.580	0.561

Table 2 (a)

	Roy equations	One loop formula
$18a_1^1 - (2a_0^0 - 5a_0^2)$	0.0410	0.0436
$18b_1^1$	0.1066	0.0753
a_2^0	15×10^{-4}	13×10^{-4}
a_2^2	$0.7 \cdot 10^{-4}$	$1.0 \cdot 10^{-4}$
$3b_0^0 - (2a_0^0 - 5a_0^2)$	0.1269	0.0989
$6b_0^2 + (2a_0^0 - 5a_0^2)$	0.1423	0.1400
a_2	0.0005	0.0005
b_0	0.030	0.027
$2a_0^0 - 5a_0^2$	0.585	0.563

Table 2 (b)

	Roy equations	One loop formula
$18a_1^1-(2a_0^0-5a_0^2)$	0.0360	0.0424
$18b_1^1$	0.1053	0.0740
a_2^0	15×10^{-4}	13×10^{-4}
a_2^2	$0.7\cdot 10^{-4}$	$1.0\cdot 10^{-4}$
$3b_0^0-(2a_0^0-5a_0^2)$	0.1200	0.1008
$6b_0^2+(2a_0^0-5a_0^2)$	0.1472	0.1407
a_2	0.0005	0.0005
b_0	0.030	0.027
$2a_0^0-5a_0^2$	0.595	0.567

Table 2 (c)

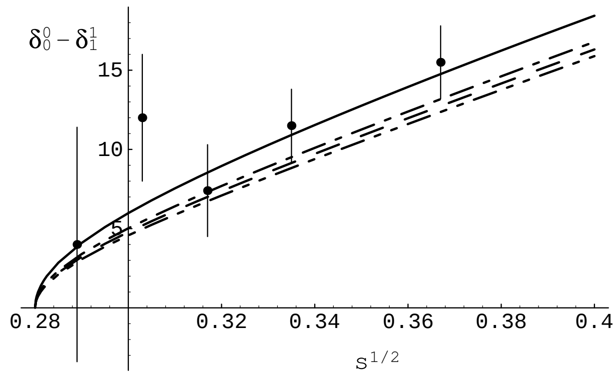

Fig. 1